MAT 219 (lecture 15)

§3.5 Undetermined Coefficients (also §4.2)

Next: More Examples (Euler Equations), Reduction of Order

§3.5 Undetermined Coeff's.

Problem: To solve

$$ay'' + by' + cy = g$$

→ variation of parameters is a lot of work (for a simple answer)

→ variation of parameters is terrible for order > 2 (i.e. $ay''' + by'' + cy' + dy = g$)

Undetermined Coefficients is a simpler method ... BUT

→ ONLY works for constant coefficients

→ ONLY works for nice functions $g(t)$

$$g = \begin{cases} \text{exponential} \\ \text{polynomial} \\ \sin / \cos \end{cases}$$ (or products of these)

Idea: $\underbrace{ay'' + by' + cy}$ $= g$

Think of this as an "operator" ← "function of functions"

$$L[f] = \underbrace{af'' + bf' + cf}$$

input is a function

output is another function

We know that solution has the form

$$y = \underbrace{c_1 y_1 + c_2 y_2} + \underbrace{Y}$$

fundamental solutions of homogeneous eqn

$$\begin{cases} L[y_1] = 0 \\ L[y_2] = 0 \end{cases}$$

one "particular" solution to non-homog. eqn

$$L[Y] = g$$

When we did variation of parameters, Y was composed of anti-derivatives

$$Y = \left(\int \frac{y_1 \cdot g}{W} dt\right) y_2 - \left(\int \frac{y_2 \cdot g}{W}\right) y_1$$

Remark: If you are bored one day, it is fun to compute $L\left[\left(\int \frac{y_1 \cdot g}{W}\,dt\right) y_2 - \left(\int \frac{y_2 \cdot g}{W}\,dt\right) y_1\right]$ using the product rule and Fund. Thm. of Calc. to see how it gives g.

Plan: Instead of computing anti-derivatives to get Y, let's just try to guess Y... or at least, guess the form of Y.

EX: What is the form of the particular solution Y for

$$y'' + 3y' + 2y = \underbrace{4t^2 + 3}\,?$$

$L[Y] =$ polynomial of degree 2

$\Rightarrow Y$ might be a polynomial of degree 2

$$\boxed{Y = At^2 + Bt + C}$$

→ Let's solve the problem completely.

EX: Solve $y'' + 3y' + 2y = 4t^2 + 2$

char. eqn.

$$r^2 + 3r + 2 = 0$$
$$(r+2)(r+1) = 0 \qquad r = -1, -2$$

$y_1 = e^{-t}$ and $y_2 = e^{-2t}$

General solution is

$$y = c_1 e^{-t} + c_2 e^{-2t} + Y$$

$$\begin{aligned}
2 \cdot Y &= 2At^2 + 2Bt + 2C \\
3 \cdot Y' &= 3 \cdot 2At + 3 \cdot B \\
+\ 1 \cdot Y'' &= 1 \cdot 2A
\end{aligned}$$

plug into DE & solve for A, B, C

$$L[Y] = 2At^2 + (6A + 2B)t + (2A + 3B + 2C)$$

$\parallel$

$4t^2 + 2$

To make these polynomials equal, we equate coefficients:

(t^2) $\quad 4 = 2A \quad \rightarrow \quad \underline{A = 2}$

(t) $\quad 0 = 6A + 2B \quad \rightarrow \quad 0 = 6 \cdot 2 + 2B$, $\quad \underline{B = -6}$

(t^0) $\quad 2 = 2A + 3B + 2C \quad \rightarrow \quad 2 = 2 \cdot 2 + 3(-6) + 2C$, $\quad \underline{C = 8}$

$$\boxed{y = c_1 e^{-t} + c_2 e^{-2t} + (2t^2 - 6t + 8)}$$

EX: Solve $y'' + 3y' + 2y = 20\cos 2t$

$L[Y] = 20\cos 2t$

General solution is

$$y = c_1 e^{-t} + c_2 e^{-2t} + Y$$

$$\begin{aligned} 2\cdot Y &= 2\cdot A\cos 2t + 2\cdot B\sin 2t \\ 3\cdot Y' &= 3\cdot 2B\cos 2t - 3\cdot 2A\sin 2t \\ +\ 1\cdot Y'' &= -4A\cos 2t - 4B\sin 2t \end{aligned}$$

Because $\frac{d}{dt}\cos t = \sin t$, $L[\cos 2t]$ will have a $\sin 2t$ term! $\Rightarrow Y$ **must** include a $\sin 2t$ term too

$$L[Y] = (-2A + 6B)\cos 2t + (-6A - 2B)\sin 2t$$

$$\| \quad 20\cos 2t$$

Equate coefficients of $\sin 2t$ and $\cos 2t$:

$(\cos 2t)$ $\quad 20 = -2A + 6B$

$(\sin 2t)$ $\quad +3(0 = -6A - 2B)$

$20 = -20A \rightsquigarrow \underline{A = -1}$

$0 = -6(-1) - 2B \rightsquigarrow \underline{B = 3}$

$$\boxed{y = c_1 e^{-t} + c_2 e^{-2t} + (-\cos 2t + 3\sin 2t)}$$

Note: Using variation of parameters would have required $\int e^{2t}\cos 2t\,dt$ and $\int e^{t}\cos 2t\,dt$ (circular integration by parts)

EX: Solve $y'' + 3y' + 2y = 24t^2e^{2t} + 16te^{2t}$

$L[Y] = (24t^2 + 16t)e^{2t}$

General Solution is

$$y = c_1 e^{-t} + c_2 e^{-2t} + Y$$

Try: $Y = \begin{pmatrix}\text{polynomial}\\ \text{of deg } 2\end{pmatrix}\cdot e^{2t}$

$$\begin{aligned} 2\cdot Y &= 2\cdot At^2e^{2t} + 2\cdot Bte^{2t} + 2\cdot Ce^{2t} \\ 3\cdot Y' &= 3\cdot 2At^2e^{2t} + 3\cdot(2A+2B)te^{2t} + 3\cdot(B+2C)e^{2t} \\ +\ 1\cdot Y'' &= 2(2A)t^2e^{2t} + \begin{pmatrix}2(2A)\\ +2(2A+2B)\end{pmatrix}te^{2t} + \begin{pmatrix}(2A+2B)\\ +2(B+2C)\end{pmatrix}e^{2t} \end{aligned}$$

$$L[Y] = 12At^2e^{2t} + (14A + 12B)te^{2t} + \begin{pmatrix}2A + 7B\\ +12C\end{pmatrix}e^{2t}$$

$$= 24t^2e^{2t} + 16te^{2t}$$

Equate coefficients of t^2e^{2t}, te^{2t}, e^{2t}:

(t^2e^{2t}) $\quad 24 = 12A \Rightarrow \underline{A = 2}$

(te^{2t}) $\quad 16 = 14A + 12B \Rightarrow 16 = 14\cdot 2 + 12B \Rightarrow \underline{B = -1}$

(e^{2t}) $\quad 0 = 2A + 7B + 12C \Rightarrow 0 = 2\cdot 2 + 7(-1) + 12C \Rightarrow \underline{C = -1/4}$

Solution:

$$\boxed{y = c_1 e^{-t} + c_2 e^{-2t} + \left(2t^2e^{2t} - te^{2t} - \tfrac{1}{4}e^{2t}\right)}$$

Note: Work is simpler if you arrange computations of Y' and Y'' into <u>columns</u>

EX: $Y = A t^2 e^{-3t} + B t e^{-3t} + C e^{-3t}$

$Y' = -3A t^2 e^{-3t} + \boxed{\cdots} t e^{-3t} + \boxed{\cdots} e^{-3t}$

$Y'' = 9A t^2 e^{-3t} + \boxed{\cdots} t e^{-3t} + \boxed{\cdots} e^{-3t}$

$t^2 e^{-3t}$ $\quad t e^{-3t}$ $\quad e^{-3t}$

EX: $Y = A t\cos 2t + B t\sin 2t + C\cos 2t + D\sin 2t$

$Y' = 2B t\cos 2t - 2A t\sin 2t + \boxed{\cdots}\cos 2t + \boxed{\cdots}\sin 2t$

$Y'' = \boxed{\cdots} t\cos 2t + \boxed{\cdots} t\sin 2t + \boxed{\cdots}\cos 2t + \boxed{\cdots}\sin 2t$

$t\cos 2t$ $\quad t\sin 2t$ $\quad \cos 2t$ $\quad \sin 2t$

Note: Computation of Y' tells how to compute Y''

EX: $Y = A t e^{2t}\cos 3t + B t e^{2t}\sin 3t + C e^{2t}\cos 3t + \cdots$

$\times 2$ $\quad \times 3$

$Y' = \begin{pmatrix} 2A \\ +3B \end{pmatrix} t e^{2t}\cos 3t + \begin{pmatrix} 2B \\ -3A \end{pmatrix} t e^{2t}\sin 3t + \begin{pmatrix} A \\ +2C \\ -3D \end{pmatrix} e^{2t}\cos 3t + \cdots$

$\times 2$ $\quad \times 3$

$Y'' = \begin{pmatrix} 2(2A+3B) \\ +3(2B-3A) \end{pmatrix} t e^{2t}\cos 3t + \begin{pmatrix} 2(2B-3A) \\ -3(2A+3B) \end{pmatrix} t e^{2t}\sin 3t + \cdots$

(You can think of Y' as $M\begin{bmatrix} A \\ B \\ \vdots \end{bmatrix}$
$\rightsquigarrow$ then Y'' is $M^2\begin{bmatrix} A \\ B \\ \vdots \end{bmatrix}$)

General Rule to guess Y from $g(t)$:

- $g(t) = \text{polynomial} \longrightarrow Y = \text{polynomial}$
- $g(t) = (\text{polynom.})e^t \longrightarrow Y = (\text{polynom})\cdot e^t$
- $g(t) = (\text{polynom.})(\sin \text{ or } \cos) \longrightarrow Y = (\text{polynom})\cdot\cos + (\text{polynom})\cdot\sin$
- etc.

Simple formulation:

Y has general terms matching (highest order terms of) $g(t)$ as well as

- $\rightarrow$ any term with <u>cos</u> must be written also with <u>sin</u>
- $\rightarrow$ any term with <u>sin</u> must be written also with <u>cos</u>
- $\rightarrow$ any term with t^n must be written also with t^{n-1} (and $t^{n-2}, \ldots, t, 1$)

EX: Solve $y'' + 3y' + 2y = 10\,t\sin t$

General solution is

$$y = c_1e^{-t} + c_2e^{-2t} + Y$$

$$2\cdot Y = 2\cdot At\sin t + 2\cdot Bt\cos t + 2C\sin t + 2D\cos t$$

$$3\cdot Y' = 3\cdot(-B)t\sin t + 3\cdot At\cos t + 3\begin{pmatrix}A\\-D\end{pmatrix}\sin t + 3\begin{pmatrix}B\\+C\end{pmatrix}\cos t$$

$$+\ Y'' = -At\sin t - Bt\cos t + \begin{pmatrix}(-B)\\-(B+C)\end{pmatrix}\sin t + \begin{pmatrix}A\\+(A-D)\end{pmatrix}\cos t$$

$$L[Y] = (A-3B)t\sin t + (3A-2B+C-3D)\sin t + (3A+B)t\cos t + (2A+3B+3C+D)\cos t$$

$$\| $$

$$10\,t\sin t$$

[Equate coefficients of $t\sin t,\ t\cos t,\ \sin t,\ \cos t$:

$(t\sin t)\quad 10 = A - 3B$

$(t\cos t)\quad +3(0 = 3A + B)$

$10 = 10A \rightarrow A = 1$

$0 = 3\cdot 1 + B \rightarrow B = -3$

$(\sin t)\quad 0 = 3A - 2B + C - 3D$

$(\cos t)\quad 0 = 2A + 3B + 3C + D$

$\boxed{A=1,\ B=-3}$

$-9 = C - 3D$

$+3(7 = 3C + D)$

$12 = 10C$

$\rightarrow C = 6/5$ and $D = 17/5$

Solution:

$$y = c_1e^{-t} + c_2e^{-2t} + t\sin t - 3t\cos t + \tfrac{6}{5}\sin t + \tfrac{17}{5}\cos t$$

EX: Solve $y'' + 3y' + 2y = 20te^t\sin t$

(This is going to get ugly...)

$$2Y = 2A\,te^t\cos t + 2C\,e^t\cos t + 2B\,te^t\sin t + 2D\,e^t\sin t$$

$$3Y' = 3(A+B)te^t\cos t + 3(A+C+D)e^t\cos t + 3(-A+B)te^t\sin t + 3(B-C+D)e^t\sin t$$

$$+\ Y'' = \begin{pmatrix}A+B\\+(-A+B)\end{pmatrix}te^t\cos t + \begin{pmatrix}(A+B)\\+(A+C+D)\\+(B-C+D)\end{pmatrix}e^t\cos t + \begin{pmatrix}-(A+B)\\+(-A+B)\end{pmatrix}te^t\sin t + \begin{pmatrix}(-A+B)\\-(A+C+D)\\+(B-C+D)\end{pmatrix}e^t\sin t$$

$$L[Y] = (5A+5B)\,te^t\cos t + (5A+2B+5C+5D)e^t\cos t + (-5A+5B)te^t\sin t + (-2A+5B-5C+5D)\,e^t\sin t$$

$$\|$$

$$20te^t\sin t$$

[Equate coefficients:

$(te^t\cos t)\quad 0 = 5A+5B$

$(te^t\sin t) + 20 = -5A+5B$

$20 = 10B \rightarrow B = 2$

$\rightarrow A = -2$

$(e^t\cos t)\quad 0 = 5A+2B+5C+5D$

$(e^t\sin t)\quad 0 = -2A+5B-5C+5D$

plug in

$6 = 5C + 5D$

$+\ 14 = -5C + 5D$

$20 = 10D \rightarrow D = 2$

$\rightarrow C = -4/5$

Solution:

$$y = c_1e^{-t} + c_2e^{-2t} - 2te^t\cos t + 2te^t\sin t - \tfrac{4}{5}e^t\cos t + 2e^t\sin t$$

There is one more important twist:

Shifting Y

EX: Solve $y'' + 3y' + 2y = te^{-t}$

General Solution

$$y = c_1 e^{-t} + c_2 e^{-2t} + Y$$

Plugging in $Y = Ate^{-t} + Be^{-t}$ does not work...

$$2 \cdot Y = 2Ate^{-t} + 2Be^{-t}$$
$$3 \cdot Y' = 3 \cdot -Ate^{-t} + 3(A-B)e^{-t}$$
$$+ \quad Y'' = Ate^{-t} + (-6A+3B)e^{-t}$$
$$L[Y] = 0 \quad - 3Ae^{-t}$$

!!! No B???

Notes: Highest order term te^{-t} vanished.

→ it is $tL[e^{-t}] = t \cdot 0$ because e^{-t} is fundamental soln.

All of the B coefficients disappeared.

→ $L[Be^{-t}] = 0$ because e^{-t} is fundamental soln.

① To get a te^{-t} term in $L[Y]$ you must have $Y = At^2e^{-t} + \cdots$

② Since e^{-t} is a fundamental solution, Y should not have an e^{-t} term

(EX continued)

→ Y should be shifted (once) because e^{-t} is a fund. soln (once)

$$Y = (Ate^{-t} + Be^{t})t$$

NO e^{-t} term

$$2 \cdot Y = 2 \cdot At^2e^{-t} + 2 \cdot Bte^{-t}$$
$$3 \cdot Y' = 3 \cdot -At^2e^{-t} + 3(2A-B)te^{-t} + 3Be^{-t}$$
$$+ \quad Y'' = At^2e^{-t} + (-4A+B)te^{-t} + (2A-2B)e^{-t}$$
$$L[Y] = \boxed{0\, t^2e^{-t}} + 2A\, te^{-t} + (2A+B)e^{-t}$$
$$\| \quad te^{-t}$$

highest order term must be 0 because e^{-t} is fund. soln.

(te^{-t}) $\quad 1 = 2A \rightsquigarrow A = 1/2$

(e^{-t}) $\quad 0 = 2A + B \rightsquigarrow B = -1$

Solution:

$$\boxed{y = c_1e^{-t} + c_2e^{-2t} + \left(\tfrac{1}{2}t^2e^{-t} - te^{-t}\right)}$$

alt. organization

$$y = \left(\tfrac{1}{2}t^2 - t + c_1\right)e^{-t} + c_2e^{-2t}$$

You must shift guesses when they contain the fund. solns. y_1 and/or y_2

EX: Solve $y'' - 2y' + y = 12te^t + e^t$

Char. eqn: $r^2 - 2r + 1 = 0$

$(r-1)^2 = 0 \qquad r = 1, 1$

General Soln is $y = c_1 e^t + c_2 t e^t + Y$

($\to e^t$ is a fundamental solution — twice

$\Longrightarrow e^t$ terms in Y must be shifted — twice)

$Y = (Ate^t + Be^t)\,t^2$

$Y = At^3e^t + Bt^2e^t$

te^t and e^t are both fundamentel solutions

$-2Y' = -2At^3e^t + -2(3A+B)t^2e^t + -2\,2Bte^t$

$+\; Y'' = At^3e^t + (6A+B)t^2e^t + (6A+4B)te^t + 2Be^t$

$L[Y] = 0t^3e^t + 0t^2e^t + 6Ate^t + 2Be^t$

$\|$

$12te^t + e^t$

TWO highest order e^t terms must be 0 because e^t is a fund. soln. TWICE

(EX continued)

$(te^t) \quad 12 = 6A \to A = 2$

$(e^t) \quad 1 = 2B \to B = 1/2$

Solution:

$$\boxed{y = c_1e^t + c_2te^t + (2t^3e^t + \tfrac{1}{2}t^2e^t)}$$

alternate organization:

$$y = (c_1 + c_2t + \tfrac{1}{2}t^2 + 2t^3)\,e^t$$

DO NOT shift parts of Y that don't include a fundamental soln

EX: What is the form of the particular solution Y for the DE

$$y'' - 4y' + 3y = te^{3t} + \cos 2t \;?$$

Char. eqn: $r^2 - 4r + 3 = 0$

$(r-3)(r-1) = 0 \qquad r = 3, 1$

Fund. solns: $y_1 = e^t$ and $y_2 = e^{3t}$

$$\boxed{Y = \underbrace{(Ate^{3t} + Be^{3t})t}_{te^{3t}} + \underbrace{C\cos 2t + D\sin 2t}_{\cos 2t}}$$

$g = te^{3t} + \cos 2t$